Error Analysis and Comparative Study of Numerical Methods for the Parabolic Equation Applied to Tunnel Propagation Modeling

Xingqi Zhang, Student Member, IEEE, and Costas D. Sarris, Senior Member, IEEE

Abstract—Parabolic equation (PE) methods have been widely applied to the modeling of wireless propagation in tunnel environments. However, the relevant literature does not include concrete guidelines for the choice of the parameters of these methods and the tradeoffs involved. This paper provides a comprehensive analysis of the two sources of error that arise when PE methods are employed for the modeling of radio-wave propagation scenarios: the well-known numerical dispersion error stemming from the finite-difference solvers for PE and the approximation error stemming from the use of PE for the solution of wave propagation problems that are subject to Maxwell’s equations. The analysis is performed for four methods, three of which have been already used in PE-based propagation studies, namely, the Crank–Nicolson (CN) scheme, the alternative-direction-implicit (ADI) method, and its locally one-dimensional (LOD-ADI) version. The fourth method is the Mitchell–Fairweather (MF)-ADI scheme that has been recently shown to be a promising alternative technique for tunnel propagation modeling. The proposed method leads to robust criteria for the choice of spatial discretization in realistic propagation scenarios, as shown via numerical examples.

Index Terms—Alternative-direction-implicit (ADI), Crank–Nicolson (CN), dispersion analysis, electromagnetic propagation, Helmholtz equation, locally one-dimensional (LOD), parabolic equation.

I. INTRODUCTION

Radio wave propagation in tunnel environments has been studied theoretically, numerically, and experimentally for several years [1]–[3]. Recently, this topic has attracted renewed interest due to the rapid development of modern communication-based train control (CBTC) systems. The deployment of access points for such systems requires knowledge of the fading characteristics of open guideway and closed tunnel sections as communication channels. While a propagation model can be obtained by measurement, the development of theoretical propagation models is highly valuable. Such models can significantly reduce the time and effort of collecting measurement data, mostly for the purpose of calibrating the model.

Several techniques have been applied to modeling radio-wave propagation in tunnels, each one presenting its own advantages and disadvantages. On the analytical side, waveguide mode theory has been widely employed [4] for fast, physically insightful yet approximate analysis of propagation along uniform tunnel geometries. The accuracy of this approach is compromised as practical tunnel cross-section geometries differ from those of standard waveguides (since they are not perfectly rectangular, circular, or elliptical), perturbing their associated mode properties. A more versatile yet computationally intensive alternative is offered by ray-tracing methods [5], [6], whose load increases with the number of reflected rays that are involved with long-distance propagation. Moreover, the underlying assumption of ray-tracing ($\lambda \rightarrow 0$) limits its accuracy in many critical cases.

Striking a better balance between accuracy and efficiency, parabolic equation (PE) techniques have found a prolific area of application in tunnel propagation studies [7]–[9]. Even in its analytical form, without considering its numerical solution, PE is a paraxial approximation to the wave equation. Therefore, it carries an approximation error, which is different from the numerical dispersion error that is commonly studied when finite-difference solvers for parabolic equations are considered. A study of how this error manifests itself, in addition to the numerical dispersion error, is shown in this paper. In particular, three commonly used, second-order accurate and unconditionally stable techniques are compared: the Crank–Nicolson (CN) [2], the Peaceman–Rachford (PR)-alternating-direction-implicit (ADI), and locally one-dimensional (LOD)-ADI method [13]. While CN is a widely used scheme, its computational cost grows significantly with problem size. Then, splitting the CN scheme into two steps, one explicit and one implicit, each containing first-order spatial finite-differences only, improves the efficiency of the method. The resulting scheme is the ADI method, which can also be made “LOD” if all spatial derivatives included in each step are taken with respect to one-dimension (1-D) only. Furthermore, a higher order accurate Mitchell–Fairweather (MF)-ADI is considered [14], [15]. This scheme is second-order accurate in the propagation direction but fourth-order accurate in the transverse directions, yet has similar computational cost as the PR-ADI and LOD methods. Applied to tunnel propagation problems, it was recently demonstrated as a sound alternative to the conventional PR-ADI, as it can attain better accuracy without additional memory or execution time requirements [16].

With this multitude of numerical methods for PE problems, a question that naturally arises is what is their relative performance and how can their parameters be predetermined...
in order to control the numerical errors in the solution, coming both from numerical dispersion and from the PE approximation of the wave equation. Surprisingly, the existing literature contains little more than mere rules of thumb with regards to this question. This paper presents a systematic study of the numerical error and the relative performance of the aforementioned four methods. As a result, specific guidelines on how to choose their discretization parameters are formulated.

The outline of this paper is as follows. First, a brief overview of the aforementioned four numerical methods for parabolic equations is given in Section II. Then, in Section III, a comprehensive analysis of the numerical error of these methods and concrete guidelines for the choice of their parameters are provided. In Sections IV and V, these guidelines are tested in specific tunnel cases and successfully compared to available measured data, leading to Section V where the main findings of this work are summarized.

II. OVERVIEW OF NUMERICAL METHODS FOR THE THREE-DIMENSIONAL (3-D) PARABOLIC EQUATION

The parabolic equation is an approximation of the Helmholtz wave equation for free-space

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2 \right) \phi(x, y, z) = 0 \]  

(1)

where \( \phi \) is a scalar potential and \( k_0 \) is the free-space wave number. Assuming propagation predominantly along the \( z \)-axis, a solution to (1) can be cast in the form

\[ \phi(x, y, z) = u(x, y, z) e^{-j k_0 z}. \]  

(2)

Assuming that \[ [\frac{\partial u^2}{\partial z^2}] \ll k_0 [\frac{\partial u}{\partial z}] \]  

(3)

which physically corresponds to paraxial propagation, and substituting (2) into (1), the standard parabolic equation is obtained

\[ \frac{\partial u}{\partial z} = \frac{1}{2j k_0} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u. \]  

(4)

The parabolic equation reduces the second-order derivative with respect to the direction of propagation (here, the \( z \)-direction) into a first-order derivative. This greatly reduces the computational cost of solving large-scale propagation problems and allows for the application of numerical methods for parabolic equations, including unconditionally stable solvers.

A. Crank–Nicolson Method

The CN method is unconditionally stable and second-order accurate in all spatial variables \( x, y, z \). The standard CN scheme for (4), shown in (5) at the bottom of the page, which can be simplified as \[ [2]\]

\[ \begin{align*}
1 - \frac{r_x}{4j k_0} \delta_x + \frac{r_y}{4j k_0} \delta_y & \quad u_{i,j}^{k+1} \\
1 + \frac{r_x}{4j k_0} \delta_x + \frac{r_y}{4j k_0} \delta_y & \quad u_{i,j}^k \\
\mathcal{O}(\Delta z^2) + \mathcal{O}(\Delta z \Delta u^2) + \mathcal{O}(\Delta z \Delta y^2) & \quad
\end{align*} \]  

(6)

where the following finite-difference approximations have been used for the second-order derivatives of (4)

\[ \begin{align*}
\delta_x u_{i,j}^k &= u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k \\
\delta_y u_{i,j}^k &= u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k
\end{align*} \]  

(7)

and \( r_x = \Delta z / \Delta x^2, r_y = \Delta z / \Delta y^2 \).

The CN scheme is shown in Fig. 1, where the dots represent the nodes that are invoked in the finite differences of (6) at the \( k \)th step.

Since the update matrix for the CN method is not tridiagonal, the computational cost of this method becomes significantly high in electrically large problems, motivating the investigation on more efficient alternative techniques.

B. PR-ADI Method

The ADI method is a modification of the CN method that has been recently introduced to the PE-based modeling of tunnel propagation in [8]. It is also unconditionally stable and

\[ \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta z} = \frac{1}{2j k_0} \left( \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{2 \Delta x^2} + \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{2 \Delta y^2} \right) \\
+ \frac{1}{2j k_0} \left( \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{2 \Delta y^2} + \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{2 \Delta y^2} \right). \]  

(5)
second-order accurate in $x$-, $y$-, and $z$-directions. However, it is more efficient because multi-dimensional fields are updated along 1-D at a time. The ADI discretization for the parabolic equation can be obtained directly by adding $(r_x r_y/(4j k_0)^2)\delta_x \delta_y$ to (6), resulting in

$$
\begin{align*}
(1 - \frac{r_x}{4 j k_0} \delta_x) (1 - \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^{k+1} &= (1 + \frac{r_x}{4 j k_0} \delta_x) (1 + \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^k \\
&+ O(\Delta z^3) + O(\Delta z \Delta x^2) + O(\Delta z \Delta y^2) + O(\Delta z^3).
\end{align*}
$$

\tag{9}

The ADI method introduces a new error term $O(\Delta z^3)$, which is of the same order as the other three error terms. The truncation error for these methods will be compared in detail in Section III.

Also, (9) can be split into two steps to obtain the fields for each direction separately. The commonly used splitting technique was first proposed by Peaceman and Rachford [11], which is also known as the Peaceman–Rachford (PR)-ADI method

$$
\begin{align*}
(1 - \frac{r_x}{4 j k_0} \delta_x) u_{i,j}^{k+\frac{1}{2}} &= (1 + \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^k \\
(1 - \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^{k+1} &= (1 + \frac{r_x}{4 j k_0} \delta_x) u_{i,j}^{k+\frac{1}{2}}
\end{align*}
$$

\tag{10}

where the index $k + 1/2$ corresponds to an intermediate plane introduced between the $z = k \Delta z$ and $z = (k + 1) \Delta z$ planes.

Note that (10a) is implicit in the $x$-direction and explicit in the $y$-direction, while (10b) is implicit in the $y$-direction and explicit in the $x$-direction, as illustrated in Fig. 2. The element values on the intermediate plane are updated line by line along the $x$-direction. After all the element values are generated for the intermediate plane, then the element values on the $z = (k + 1) \Delta z$ plane are updated line by line along the $y$-direction. The PR-ADI method uses smaller tridiagonal matrices, hence significantly improving upon the efficiency of the CN technique.

C. LOD-ADI Method

An alternative way to split (9) is the LOD method [13]. The LOD-ADI update equations are similar to the PR-ADI method; however, each substep includes spatial finite-differences with respect to either $x$ or $y$ only. The update equations are

$$
\begin{align*}
(1 - \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^{k+\frac{1}{2}} &= (1 + \frac{r_x}{4 j k_0} \delta_x) u_{i,j}^k \\
(1 - \frac{r_x}{4 j k_0} \delta_x) u_{i,j}^{k+1} &= (1 + \frac{r_y}{4 j k_0} \delta_y) u_{i,j}^{k+\frac{1}{2}}
\end{align*}
$$

\tag{11}

Evidently, (11a) includes only finite differences with respect to $y$, whereas (11b) includes only finite differences with respect to $x$. This “LOD” nature of the method is depicted in Fig. 3.

D. MF-ADI Method

Instead of the second-order approximation given in (7), MF-ADI employs the following fourth-order approximation for the second-order derivatives $\partial^2 u/\partial \xi^2$, $\xi = x, y$ with respect to the transverse plane coordinates. The update equations for this method are [15], [16]

$$
\begin{align*}
\left[1 - \left(\frac{r_x}{4 j k_0} - \frac{1}{12}\right) \delta_x\right] u_{i,j}^{k+\frac{1}{2}} &= \left[1 + \left(\frac{r_y}{4 j k_0} + \frac{1}{12}\right) \delta_y\right] u_{i,j}^k \\
\left[1 - \left(\frac{r_y}{4 j k_0} - \frac{1}{12}\right) \delta_y\right] u_{i,j}^{k+1} &= \left[1 + \left(\frac{r_x}{4 j k_0} + \frac{1}{12}\right) \delta_x\right] u_{i,j}^{k+\frac{1}{2}}
\end{align*}
$$

\tag{12}

This scheme is known as the Mitchell–Fairweather (MF)-ADI method. The updating procedure of the fields is the same as shown in Fig. 2. Similar to the PR-ADI method, the update equations invoke tridiagonal matrices and the computational cost remains the same. However, the MF-ADI method is of higher order accuracy: second-order accurate in $z$-direction and fourth-order accurate in $x$- and $y$-directions. Hence, the improvement in the order of accuracy is achieved without any overhead on operation count.
III. COMPARATIVE ERROR ANALYSIS OF CN, PR-ADI, LOD-ADI, AND MF-ADI METHODS FOR THE PARABOLIC EQUATION

In this section, the numerical errors involved in the application of the aforementioned four numerical schemes to wave propagation problems are analyzed and compared. This analysis is divided into two parts. First, an error analysis which includes both the standard dispersion error of parabolic solvers and the error introduced by the PE approximation of the wave equation is presented. The results of this analysis can indeed guide the choice of the cell sizes $\Delta x$, $\Delta y$, and $\Delta z$ for each scheme, as well as illustrate their relative performance. For completeness, the standard numerical dispersion analysis for parabolic equation solvers (i.e., one that simply focuses on the numerical dispersion error, disregarding the approximation error) is presented as well.

A. Complete Error Analysis

Consider an exact solution of the Helmholtz wave equation (1) with respect to a scalar potential $\phi$

$$\phi(x, y, z) = C e^{-j(k_x x+k_y y+k_z z)}$$  \hspace{1cm} (13)

where $C$ is a constant coefficient. Then, $u(x, y, z)$ in (2) is obtained as

$$u(x, y, z) = C e^{-j[k_x x+k_y y+(k_z-k_0)z]}$$  \hspace{1cm} (14)

where $k_0$ is given by the consistency condition

$$k_0^2 = k_x^2 + k_y^2 + k_z^2.$$  \hspace{1cm} (15)

By substituting the plane-wave solution (14), in a discrete-space form, into the four numerical schemes for the parabolic equation, the numerical wavenumber $\tilde{k}_0$ can be extracted. Note that the components of the numerical wavevector at a direction defined by the angles $(\varphi, \theta)$ of the spherical coordinate system are given by

$$\tilde{k}_x = \tilde{k}_0 \cos \varphi \sin \theta, \quad \tilde{k}_y = \tilde{k}_0 \sin \varphi \sin \theta, \quad \tilde{k}_z = \tilde{k}_0 \cos \theta.$$  \hspace{1cm} (16)

Then, the deviation of the numerical wavenumber $\tilde{k}_0$ from the actual wavenumber $k_0$ can be determined as a function of the cell size. In the following, this calculation is performed for each of the four numerical methods under consideration.

1) Crank–Nicolson Method: Substituting (14) into (6), the following numerical dispersion relation for the CN method is obtained

$$\tan \left( \frac{k_0 - k_z}{2} \right) \Delta z = \frac{\Delta z}{k_0 \Delta x^2} \sin^2 \frac{k_x \Delta x}{2} + \frac{\Delta z}{k_0 \Delta y^2} \sin^2 \frac{k_y \Delta y}{2} \frac{k_0 \Delta x}{2}.$$  \hspace{1cm} (17)

Letting $\Delta x = \Delta y$, $\alpha = \Delta z / (k_0 \Delta x^2)$, and $N_\lambda = \lambda / \Delta x = 2\pi / (k_0 \Delta x)$ (where $N_\lambda$ is a parameter controlling the number of nodes on the cross-section, as well as along the direction of propagation, per wavelength) and substituting (16) into (17), then the following equation is derived, with respect to the normalized numerical wavenumber $\tilde{k}_0 \lambda$

$$\tan \left( \frac{\tilde{k}_0 \lambda (1 - \cos \theta) \alpha \pi}{N_\lambda^2} \right) = \frac{\alpha \sin^2 \tilde{k}_0 \lambda \sin \varphi \sin \theta}{2N_\lambda} \frac{\tilde{k}_0 \lambda \cos \varphi \sin \theta}{2N_\lambda}.$$  \hspace{1cm} (18)

This nonlinear equation is easily solved via the Newton–Raphson method. Since analytically $k_0 \lambda = 2\pi$, the absolute value of the relative error in the numerical wavenumber is

$$E = |\tilde{k}_0 - k_0| / k_0.$$  \hspace{1cm} (19)

Obviously, $E$ produces a phase and magnitude error of the propagating wave along the direction of propagation. This metric is used as an indicator of the relative accuracy of the four methods under comparison.

2) PR-ADI Method: Repeating the process of the previous section for the PR-ADI method, (14) is entered into (10a) and (10b). Then, the following numerical dispersion relations are obtained for the PR-ADI substeps:

$$e^{-j(k_z - k_0)\Delta z_1} = \frac{1 + j \frac{\Delta z}{k_0 \Delta y} \sin^2 \frac{k_y \Delta y}{2}}{1 - j \frac{\Delta z}{k_0 \Delta y} \sin^2 \frac{k_y \Delta y}{2}}$$ \hspace{1cm} (20a)

$$e^{-j(k_z - k_0)\Delta z_2} = \frac{1 + j \frac{\Delta z}{k_0 \Delta y} \sin^2 \frac{k_y \Delta y}{2}}{1 - j \frac{\Delta z}{k_0 \Delta y} \sin^2 \frac{k_y \Delta y}{2}}$$ \hspace{1cm} (20b)

where $\Delta z_1 = \Delta z_2 = \Delta z / 2$. Further through similar steps, the normalized wavenumbers for the two substeps $\tilde{k}_{0,1} \lambda$ and $\tilde{k}_{0,2} \lambda$ are derived through the following expressions:

$$\tilde{k}_{0,1} \lambda (1 - \cos \theta) \alpha \pi \frac{e}{N_\lambda^2} = \frac{1 + j \alpha \sin^2 \tilde{k}_{0,1} \lambda \sin \varphi \sin \theta}{2N_\lambda}$$ \hspace{1cm} (21a)

$$\tilde{k}_{0,2} \lambda (1 - \cos \theta) \alpha \pi \frac{e}{N_\lambda^2} = \frac{1 + j \alpha \sin^2 \tilde{k}_{0,2} \lambda \cos \varphi \sin \theta}{2N_\lambda}$$ \hspace{1cm} (21b)

3) LOD-ADI Method: The numerical dispersion relation for the LOD method can be obtained by exchanging the denominators of the right-hand side of (20a) and (20b), yielding the following equations for the normalized wavenumbers of the two substeps (following the same notation as in the PR-ADI case)

$$\tilde{j}k_{0,1} \lambda (1 - \cos \theta) \alpha \pi \frac{e}{N_\lambda^2} = \frac{1 + j \alpha \sin^2 \tilde{j}k_{0,1} \lambda \sin \varphi \sin \theta}{2N_\lambda}$$ \hspace{1cm} (22a)

$$\tilde{j}k_{0,2} \lambda (1 - \cos \theta) \alpha \pi \frac{e}{N_\lambda^2} = \frac{1 + j \alpha \sin^2 \tilde{j}k_{0,2} \lambda \cos \varphi \sin \theta}{2N_\lambda}$$ \hspace{1cm} (22b)
Fig. 4: Comparison of dispersion phase error per $\lambda$ versus discretization between different numerical schemes.

4) MF-ADI Method: Substituting (14) into (12a) and (12b), the equations for the normalized wavenumbers of the two substeps can be derived

$$
\frac{j\tilde{k}_{0,1}\lambda(1 - \cos \theta)}{e} \frac{(1 - \frac{j\alpha}{3})}{N_{\lambda}^2} = 1 + \frac{(j\alpha - \frac{1}{3})}{1 - (j\alpha + \frac{1}{3})} \sin^2 \frac{k_{0,1}\lambda \sin \varphi \sin \theta}{2N_{\lambda}}
$$

(23a)

$$
\frac{j\tilde{k}_{0,2}\lambda(1 - \cos \theta)}{e} \frac{(1 - \frac{j\alpha}{3})}{N_{\lambda}^2} = 1 + \frac{(j\alpha - \frac{1}{3})}{1 - (j\alpha + \frac{1}{3})} \sin^2 \frac{k_{0,2}\lambda \cos \varphi \sin \theta}{2N_{\lambda}}
$$

(23b)

Since for the PR-ADI, LOD, and MF-ADI schemes, the equations are split into two steps, the numerical phase per cell is the sum of the two steps

$$\hat{k}_0 \Delta z = (\hat{k}_{0,1} + \hat{k}_{0,2}) \Delta z / 2. \quad (24)$$

5) Numerical Results: Fig. 4 illustrates the phase error (19) for the four numerical schemes as a function of the number of points per wavelength $N_{\lambda}$ and the zenith angle $\theta$, where $\alpha = 4$ and $\varphi = 45^\circ$. It can be observed that the PR-ADI and LOD schemes have the same performance. The MF-ADI scheme has the lowest phase error, and the CN scheme has the largest phase error. Moreover, the phase error reduces as $N_{\lambda}$ increases. Asymptotically, all four methods converge to the same error floor, which represents the approximation error of the parabolic equation. This is an insightfull result that cannot be extracted via conventional dispersion analysis of parabolic equation solvers and indicates the value of the proposed approach. As expected, this floor is reduced as the paraxial propagation assumption of the parabolic equation is better met. Note, e.g., that this error floor is much lower (and close to zero) for $\theta = 8^\circ$ than for $\theta = 8^\circ$.

To better illustrate how the approximation error evolves as a function of the angle $\theta$ for a fixed finite-difference scheme and discretization rate, Fig. 5 shows the phase error with respect to $\theta$, with $N_{\lambda} = 5$, $\alpha = 4$, $\varphi = 45^\circ$. A general “rule of thumb,” which was previously stated in [2], was that the wavevector should be confined within $15^\circ$ from the $z$-axis for the PE approximation error to be acceptable. Notably, these results offer a more comprehensive, quantitative description of this error.

Fig. 6 illustrates the phase error as a function of $\alpha$, where $N_{\lambda} = 5$, $\theta = 8^\circ$, and $\varphi = 45^\circ$. It can be seen that with increasing $\alpha$, advantages for PR-ADI, LOD, and MF-ADI schemes over CN scheme become more obvious, which can be further proved by the discretization error analysis in part B. Moreover, Fig. 7 depicts the phase error as a function of the azimuth angle $\varphi$ (in deg), with $N_{\lambda} = 5$, $\alpha = 4$, and $\theta = 8^\circ$. Minimum error is observed for wave propagation along the diagonal ($\varphi \approx 45^\circ, 135^\circ$). This effect is due to the numerical dispersion error of finite differences rather than the PE approximation and is encountered in other finite-difference methods as well, such as...
the finite-difference time-domain (FDTD) technique. Assume, e.g., that a target error of $10^{-3}$ is set for a case when $\theta \leq 4^\circ$. Then, $N_{\lambda} \geq 5$ would allow the CN, PR-ADI, and LOD methods to meet the target, whereas $N_{\lambda} > 2$ would suffice for the MF-ADI. Further calibration of the error characteristics of these methods is possible by tuning $\alpha$.

B. Numerical Dispersion Error of the CN, PR-ADI, LOD-ADI, and MF-ADI Methods

For completeness, we also present the standard numerical dispersion analysis of the four PE solvers under comparison [13], which isolates their truncation error from the PE approximation error. The general form of the solution to the parabolic equation

$$u(x, y, z) = C e^{x jk_x x + jk_y y + jk_z z}$$

is now subject to the dispersion relation:

$$2k_0 k_z = k_x^2 + k_y^2 = k_t^2.$$  \hspace{1cm} (26)

The truncation errors of the four schemes, defined as in [13], are as follows:

1) **CN scheme:**

$$T_{CN} = \frac{jk_z^2 u}{96k_0^3} \Delta z^2 + \frac{jk_z^2 u}{24k_0} \Delta x^2 + \frac{jk_z^2 u}{24k_0} \Delta x^2. \hspace{1cm} (27)$$

2) **PR and LOD-ADI schemes:**

$$T_{PR/LOD-ADI} = T_{CN} - \frac{jk_z^2 k_y (k_x^2 + k_y^2)}{32k_0^2} u \Delta z^2. \hspace{1cm} (28)$$

3) **MF-ADI scheme:**

$$T_{MF-ADI} = \frac{j}{32} \left( \frac{1}{3} - \frac{k_z^2 k_y}{k_t^2} \right) \frac{k_0 u}{k_0} \Delta z^2$$

$$+ \frac{jk_z^2 u}{288k_0} \left[ k_x^2 k_y \Delta x^2 \Delta y^2 + k_z^4 \Delta x^4 + k_y^4 \Delta y^4 \right]. \hspace{1cm} (29)$$

These expressions illustrate that while the PR and LOD-ADI are improved versions of the CN method, MF-ADI is actually higher order accurate with respect to $\Delta x$, $\Delta y$, as the corresponding leading error terms in its truncation error are fourth order ($O(\Delta x^4)$, $O(\Delta y^4)$, $O(\Delta x^2 \Delta y^2)$). However, these standard results are only partially helpful for PE error analysis in propagation modeling. The following section demonstrates that the analysis of Section III-A effectively captures the dependence of the total error (numerical dispersion and approximation) that arises in finite-difference schemes for the PE, when these are applied to propagation problems.

IV. Numerical Results: Rectangular Waveguide Analysis

The purpose served by this section is twofold. First, the proposed error analysis of Section III-A is corroborated by numerical and analytical results, derived from modal analysis of a rectangular waveguide with perfect electric conductor walls. Second, the same structure is used to compare the performance of the four methods under study, for Dirichlet boundary conditions applied on the walls of the waveguide.

A. Numerical Validation of Error Analysis

The following numerical test has been designed to validate the analysis of Section III-A. A square waveguide is simulated at 2.4 GHz. This structure supports guided modes. An $(m, n)$ mode has the propagation constant

$$k_z^{(m,n)} = 2\pi \sqrt{n_0 \mu_0} \sqrt{f^2 - f_c^2)^2}$$

where

$$f_c^{(m,n)} = \frac{1}{2\pi c_0 \mu_0} \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}$$

is the cut-off frequency of the mode, and $a$ and $b$ are the width and height of the guide, respectively.

The cross-section of the waveguide is $15\lambda \times 15\lambda$ at the simulated frequency of 2.4 GHz. All simulations use a grid with $\Delta x = \Delta y = 0.3\lambda$, and $\Delta z = \lambda$. Sampling the electric field $E(x, y, z)$ along the $z$-direction and taking the Fourier transform with respect to $z$, the 1-D Fourier transform $\mathcal{F}_z\{E(x, y, z)\}$ is derived. The spectrum of $\mathcal{F}_z\{E(x, y, k_z)\}$ consists of peaks that correspond to the propagation constants of the guided modes. Hence, we can determine the cut-off frequencies $f_c^{(m,n)}$ in three ways: first, analytically, second, via the dispersion analysis of Section III-A by setting $k_x = 2\pi n/a$, $k_y = 2\pi n/b$, $k_z = 0$ and finding the frequency, and third, from the peaks of the spectrum of $\mathcal{F}_z\{E(x, y, k_z)\}$. These three sets of results are collected in Table I. Notably, theory, error analysis, and simulation results are in good agreement with each other for all the modes included in the table.

B. Comparative Performance Evaluation of CN, PR-ADI, LOD-ADI, MF-ADI for a Given Error Target

We proceed with a combined error/computation time analysis of the four methods under study. Since the rectangular
TABLE I

COMPARISON BETWEEN RESONANT FREQUENCIES (IN 10⁶ Hz) FROM THEORY, THE ERROR ANALYSIS OF SECTION III-A AND SIMULATION

<table>
<thead>
<tr>
<th>Mode</th>
<th>Theory</th>
<th>CN (Section III-A)</th>
<th>CN (Simulation)</th>
<th>PR-ADI (LOD) (Section III-A)</th>
<th>PR-ADI (LOD) (Simulation)</th>
<th>MF-ADI (Section III-A)</th>
<th>MF-ADI (Simulation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m, n)</td>
<td>fmn</td>
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<tr>
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<td>1.1309</td>
<td>1.1310</td>
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<tr>
<td>(2, 1)</td>
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<td>1.7865</td>
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<td>1.7876</td>
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<tr>
<td>(2, 2)</td>
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<td>2.2587</td>
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<tr>
<td>(3, 1)</td>
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<td>3.3805</td>
<td>3.3805</td>
<td>3.3855</td>
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</table>

waveguide admits an analytical solution, an Euclidean error norm can be defined, invoking the analytical field solutions as follows:

$$E_{rms} = \sqrt{\frac{1}{N} \sum_{i} \sum_{j} \left| u_{i,j}^{num} - u_{i,j}^{exact} \right|^2}$$

(30)

where $u^{num}$ denotes numerically and $u^{ana}$ denotes analytically computed fields, respectively; $N$ is the number of the discretization points on the cross-section at $z = k_f \Delta z$ and $k_f$ is the number of spatial steps along the propagation direction.

Fig. 8 illustrates the analytical and numerical transverse-field patterns, respectively, at $z = 100$ m (excitation plane is $z = 0$). The tunnel cross-section’s dimensions are still $15 \lambda \times 15 \lambda$ at the maximum operating frequency of 2.4 GHz, where $\Delta x = \Delta y = 0.4 \lambda$, $\alpha = 4$. A unit strength Gaussian source with $2.5 \lambda$ standard deviation is placed at the center of the $z = 0$-plane as the initial condition. With these parameters, the field solution primarily consists of low-order waveguide modes.

As can be seen, the MF-ADI method has better agreement with the analytical results, and the rms error is much smaller than the other three methods. The PR-ADI and LOD methods have the same rms error, and the CN method has the largest rms error.
Then, the field solutions obtained at $z = 100$ m by the four numerical schemes are compared with the analytical solution, obtained by the Helmholtz equation and the parabolic equation. In the former case, the error includes both approximation and dispersion error, while in the latter case, it includes only the dispersion error. Hence, the results of this analysis illustrate how these errors vary as the discretization parameters change in the four schemes.

The error norm (30) is plotted as a function of $\Delta x/\lambda$ (with $\alpha = 4$) in Fig. 9 and as a function of $\alpha$ (with $\Delta x = 0.3\lambda$) in Fig. 10. The thick lines represent the errors of these schemes with respect to the analytical solution of the wave equation. The thin lines represent errors with respect to the analytical solution of the parabolic equation.

As can be seen from Figs. 9 and 10, the MF-ADI method has the best performance; the PR-ADI and LOD methods have the same performance, and the CN method has the worst performance among the four methods. Moreover, it can be observed from Fig. 9 that the slope of the error curves for the MF-ADI method is about twice the slope of the other three methods. This agrees with the fact that the MF-ADI method is fourth-order accurate and the others are second-order accurate with respect to the transverse directions. It can also be seen that the MF-ADI method converges much faster than the others. This allows for a larger $\Delta x$ to be chosen for the MF-ADI method for the same error target.

Additionally, it can be observed from Fig. 9 that when $\Delta x$ decreases, unlike the thin lines, the rms errors for the thick lines converge to constant values instead of going to zero. This error floor, shared by all methods, is set by the approximation error of the parabolic equation. In general, all results are in agreement with the basic conclusions of our error analysis, providing an additional confirmation of its usefulness and validity.

Finally, Fig. 11 illustrates the relationship between accuracy and CPU time (in seconds) for the four methods. It can be seen that for a given computation time, the MF-ADI method is always more accurate than the other three methods. The figure also shows how the computational cost of the CN method grows as the number of cells increases.

V. APPLICATION: PATH LOSS ESTIMATION FOR THE MASSIF CENTRAL TUNNEL

In this section, the four numerical methods under study are applied to determine path loss in a tunnel geometry that has been widely studied in the literature, the Massif Central tunnel in south-central France [19]. It is a straight, 3.5-km long tunnel consisting of large blocks of smooth stones or concrete. The roughness of the walls is difficult to estimate but in the order
and simulated data are depicted in Fig. 12, where the straight lines represent the corresponding attenuation.

of centimeters. Although the tunnel cross-section is not rectangular, it can be approximated with a rectangular cross-section 7.8 m × 5.3 m, while wall parameters are set to \( \varepsilon_r = 5 \) and \( \sigma_0 = 0.01 \text{ S/m} \), following [19].

Exact analytical solutions for the fields inside rectangular tunnels with lossy walls do not exist [17], [18]. Instead, impedance boundary conditions can be applied to model lossy tunnels by treating the walls as imperfect conductors with a surface impedance [18]. Here, the Leontovich impedance boundary condition [17] is used

\[
\hat{n} \times E = -Z_s \hat{n} \times (\hat{n} \times H)
\]

(31)

where \( Z_s \) is the relative surface impedance and \( \hat{n} \) is the outward normal on the wall. For a wall with relative permittivity \( \varepsilon_r \) and conductivity \( \sigma_0 \) (in \( \text{S/m} \)), \( Z_s \) is approximated as [7]

\[
Z_s = \frac{\sqrt{\varepsilon_{rc} - 1}}{\varepsilon_{rc}}
\]

(32)

where \( \varepsilon_{rc} = \varepsilon_r - j\sigma_r \) and \( \sigma_r = \sigma_0 / \omega \varepsilon_0 \) are, respectively, the complex permittivity and relative conductivity. Notably, the use of tensor impedance boundary conditions in a vector parabolic equation method would capture wave depolarization effects that play an important role in curved tunnels [9]. However, the use of (32) was deemed sufficient in this case.

Tunnel propagation is simulated at 900 MHz along 2.5 km. For the transverse discretization, \( \Delta x \) and \( \Delta y \) are chosen as 1.5λ, and \( \Delta z = 4\lambda \). A ray-tracing method [6] is applied for the first 500 m of the tunnel, which are characterized by multimode interference. Then, PE is applied when low-order modes, satisfying its fundamental assumption, dominate.

The simulated results are compared to the measured data from [19]. The received power is recorded at height \( y = 2 \text{ m} \) at one-quarter of the width of the tunnel horizontally. Measured and simulated data are depicted in Fig. 12, where the straight lines represent the corresponding attenuation, which is calculated by a least-squares fit to the data in the PE region (0.5–2.5 km).

Table II presents the attenuation obtained from the measured and simulated data for the Massif Central tunnel. This realistic case study further confirms the superior performance of the MF-ADI method, with respect to the other three methods.

VI. CONCLUSION

Parabolic equation solvers have been developed and successfully applied to wireless propagation modeling for many years. However, the rich literature on this topic was still missing a comprehensive error analysis of established, as well as newer finite-difference schemes, such as the CN and the ADI-based ones (PR, LOD, and MF-ADI), respectively. This paper has contributed such an error analysis that allows one to choose the parameters of each method based on certain error targets. Modified numerical dispersion expressions, including both the conventional dispersion error and the PE approximation error, have been provided for the four numerical schemes that were studied. The procedure outlined in this paper is applicable to any alternative finite-difference scheme as well.

These expressions reveal that the recently introduced PR-ADI [8], as well as the MF-ADI method [16], present significant advantages compared to the CN scheme. Among the three ADI methods, MF-ADI is more accurate for the same execution time, or faster for the same accuracy, than the other two ADI methods. Hence, these comparisons lead to the conclusion that MF-ADI is the method of choice for the PE-based solution of tunnel propagation problems, because it employs fourth-order accurate finite differences over the electrically large tunnel cross-section.

REFERENCES


![Fig. 12. Performance comparison of the CN, PR-ADI, LOD, and MF-ADI methods in the Massif Central tunnel with lossy walls, where the straight lines represent the corresponding attenuation.](image-url)


