A Subgridding Approach for the FDTD Solution of the Nonuniform Transmission Line Equations Based on the Renormalization of the Per-Unit-Length Parameters

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A Subgridding Approach for the FDTD Solution of the Nonuniform Transmission Line Equations Based on the Renormalization of the Per-Unit-Length Parameters

Roberto B. Armenta and Costas D. Sarris

Abstract—One of the most challenging aspects of the finite-difference time-domain (FDTD) solution of the nonuniform transmission line (NTL) equations is to choose a discretization grid with an adequate spatial resolution. In the standard formulation of the algorithm, a uniform discretization grid is used, however, different regions of the NTL may require different spatial resolution levels. To address this issue, this paper introduces a subgridding scheme with a unique feature: it preserves the simplicity of the standard FDTD time stepping algorithm. It is shown that all the details of the fine subgrids can be absorbed into effective per-unit-length parameters through a coordinate system transformation. By using the proposed coordinate system transformation, a nonuniformly spaced grid can be mapped onto a uniformly spaced grid where the standard FDTD update equations can be applied.

Index Terms—distributed parameter circuits, finite difference methods, time domain analysis, transmission line modeling.

I. INTRODUCTION

The finite-difference time-domain (FDTD) method for solving the nonuniform transmission-line (NTL) equations is a simple, widely used and versatile tool. The standard algorithm—presented in [1]—was formulated for the case of uniform transmission lines with constant per-unit-length parameters. Nonetheless, the method can be easily extended to the analysis of NTLs with position dependent per-unit-length parameters; all that is needed is to evaluate the per-unit-length parameters at the appropriate points on the discretization grid. One of the most challenging aspects of the FDTD solution of the NTL equations is to choose a discretization grid with an adequate spatial resolution. In the original algorithm, a uniform discretization in time and space is used. However, when dealing with position dependent per-unit-length parameters, different regions of the NTL may require very different levels of spatial resolution. In such case, using a uniformly spaced discretization grid produces an unnecessary computational overhead. In the past, two different alternatives have been proposed to address this issue: the first one is to use a wavelet based multiresolution time domain algorithm (for e.g. [2], [3]); and the second one is to incorporate a subgridding scheme (for e.g. [4], [5]) into the standard FDTD algorithm.

While many successful subgridding schemes have been proposed before (see [6] for a comprehensive summary), all of them introduce modifications to the standard FDTD update equations at the interface points between grids with different resolution levels. Even though the update equations for interface points are usually straightforward modifications of the standard FDTD update equations, dealing with grid interfaces can significantly increase the complexity of implementing the modified FDTD scheme. This paper describes a subgridding scheme that preserves the simplicity of the standard FDTD time stepping algorithm. It is shown that all the details of the finer subgrids can be absorbed into effective (or renormalized) per-unit-length parameters through a one-dimensional coordinate system transformation. By using a coordinate system transformation, a nonuniformly spaced grid can be mapped onto a uniformly spaced grid where the standard FDTD update equations can be applied. The idea of using a coordinate transformation to introduce subgrids was originally suggested in [7], and it is applied here in the context of the NTL equations. The rest of this paper is divided into three sections. Section II briefly reviews the standard FDTD procedure for solving the NTL equations. Section III introduces a one-dimensional coordinate system transformation (called the step summing transformation) that can be used to map a nonuniformly spaced grid onto a uniformly spaced one. Finally, in Section IV a practical example of the proposed subgridding scheme is presented.

II. THE STANDARD FDTD ALGORITHM

For a general NTL, the position and time dependent voltage and current functions $V(z, t)$ and $I(z, t)$ are related by

$$\frac{\partial V(z, t)}{\partial t} = -\frac{G(z)}{C(z)} V(z, t) - \frac{1}{C(z)} \frac{\partial I(z, t)}{\partial z} \tag{1}$$

and

$$\frac{\partial I(z, t)}{\partial t} = -\frac{R(z)}{L(z)} I(z, t) - \frac{1}{L(z)} \frac{\partial V(z, t)}{\partial z} \tag{2}$$

gwhere $G(z)$ is the per-unit-length conductance, $R(z)$ is the per-unit-length resistance, $C(z)$ is the per-unit-length capacitance and $L(z)$ is the per-unit-length inductance of the NTL. In the standard FDTD algorithm, (1) and (2) are discretized on a uniformly spaced grid using second order accurate centered difference approximations at points $(i\Delta z, (n + 1/2)\Delta t)$ and $(i + 1/2)\Delta z, n\Delta t$, respectively. This yields the following update equations:

$$V_{i}^{n+1} = (\xi_{i}^{\pm})^{-1} \left[ \xi_{i}^{\pm} V_{i}^{n} - \left( I_{i+1/2}^{n+1/2} - I_{i-1/2}^{n+1/2} \right) \right] \tag{3a}$$

and

$$\xi_{i}^{\pm} = \left( \frac{C_{i} \Delta z}{\Delta t} \pm \frac{G_{i} \Delta z}{2} \right) \tag{3b}$$

and

$$I_{i+1/2}^{n+1/2} = \left( c_{i+1/2}^{\pm} \right)^{-1} \left[ c_{i+1/2}^{\pm} I_{i+1/2}^{n-1/2} - (V_{i+1}^{n} - V_{i}^{n}) \right] \tag{4a}$$

and

$$c_{i+1/2}^{\pm} = \left( L_{i+1/2} \Delta z / \Delta t \right) \pm \frac{R_{i+1/2} \Delta z}{2} \tag{4b}$$

for all $i = 0, 1, 2, \ldots, N_{x}$ and $n = 0, 1, 2, \ldots, N_{t}$ where

$$V_{i}^{n} = V(z = i\Delta z, t = n\Delta t), \quad C_{i} = C(z = i\Delta z);$$

$$I_{i+1/2}^{n+1/2} = I(z = (i + 1/2)\Delta z, t = (n + 1/2)\Delta t);$$

and so on. Equations (3) and (4) are solved in a leapfrog fashion assuming that $V_{0}^{0} = 0$ for all $i$, and that voltage nodes $V_{0}^{n}$ and $V_{N_{x}}^{n}$ are updated in time through a separate set of equations that arise from the boundary conditions.

A. Spatial Resolution of a Uniform Grid

When a uniform discretization grid is used, the cell size $\Delta z$, which determines the spatial resolution of the grid, is selected so that

$$\lambda_{\text{min}}/20 \leq \Delta z \leq \lambda_{\text{min}}/10 \tag{5a}$$

where

$$\lambda_{\text{min}} = \min_{z = 0}^{20 + L_{G}} \{ \lambda(z) \}, \quad \lambda(z) = \frac{1}{f_{\text{max}} \sqrt{L(z)C(z)}} \tag{5b}$$

In the above, $f_{\text{max}}$ is the maximum frequency excited by the source, and $L_{G}$ is the length of the NTL along the z-axis. Also, the time step $\Delta t$ is picked according to

$$\lambda_{\text{min}} f_{\text{max}} \Delta t/\Delta z \leq 1 \tag{6}$$

in order to satisfy stability requirements. Whenever $\Delta z$ and $\Delta t$ are picked according to (5) and (6), the uniform grid will be unnecessarily fine in regions where $\lambda(z)$ is much larger than $\lambda_{\text{min}}$. This well-known problem can be addressed by introducing localized high resolution mesh blocks known as subgrids.
parameters that correspond to each section are presented in Table I.

Fig. 1. NTL example under consideration. The NTL has been divided into five sections, which are labeled with underlined numbers. The per-unit-length parameters that correspond to each section are presented in Table I.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( L_m )</th>
<th>( C_m )</th>
<th>( m ) = Section Number</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mu_0 ) ( \varepsilon_0 )</td>
<td>( L_m ) = Per-Unit-Length Inductance</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \mu_0 ) ( 25\varepsilon_0 )</td>
<td>( C_m ) = Per-Unit-Length Capacitance</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \mu_0 ) ( \varepsilon_0 )</td>
<td>( \mu_0 = 4\times10^{-7} ) ( \text{H/m} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \mu_0 ) ( 25\varepsilon_0 )</td>
<td>( \varepsilon_0 = 8.8542 \times 10^{-12} ) ( \text{F/m} )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \mu_0 ) ( \varepsilon_0 )</td>
<td>( c_0 = 1/\sqrt{\mu_0\varepsilon_0} )</td>
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B. Traditional Subgridding

Subgrids are usually introduced in three steps. In step 1, a piecewise constant approximation of \( \lambda(z) \) is introduced. The NTL’s \( z \)-axis domain is divided into \( M \) nonoverlapping sections where the \( m \)-th section is contained within the interval \([z_{m-1}, z_m]\). Then, \( \lambda(z) \) is approximated by a constant value

\[
\lambda_m^m = \min_{z = z_{m-1}} \{ \lambda(z) \} \tag{7}
\]

inside the \( m \)-th section. In step 2, all the sections where

\[
\lambda_m^m \approx \lambda_{\text{max}}, \quad \lambda_{\text{max}} = \max_{m=1} \{ \lambda_m^m \} \tag{8}
\]

are put together into a single entity called the main grid region, which receives a coarse discretization grid. Once the main grid region has been defined, finer grids are assigned to the remaining NTL sections, which are called the subgridding regions. Finally, in step 3, a special set of equations is generated to update the grid interface points. The alternative subgridding approach proposed here consists of keeping the first two steps and replacing the last one with a coordinate transformation that maps the nonuniform grid generated in step 2 from \((z, t)\)-space to \((\bar{z}, t)\)-space. If chosen as described in the next section, the coordinate transformation can be used to map a nonuniform grid in \((z, t)\)-space to a uniform grid in \((\bar{z}, t)\)-space, where the problem can be solved using (3) and (4). (3)

III. THE COORDINATE SYSTEM TRANSFORMATION

To define a coordinate transformation \( z \rightarrow \bar{z} \), we need to specify a function \( z = f(\bar{z}) \). Substituting \( z = f(\bar{z}) \) into (1) and (2) yields

\[
\frac{\partial V(z, t)}{\partial t} = -G(z)V(z, t) - \frac{1}{C(z)} \frac{\partial I(z, t)}{\partial z} \tag{9a}
\]

\[
G(\bar{z}) = G(z)f'(\bar{z}), \quad C(\bar{z}) = C(z)f'(\bar{z}) \tag{9b}
\]

and

\[
\frac{\partial I(z, t)}{\partial t} = \frac{R(z)}{L(z)} f(z, t) - \frac{1}{L(z)} \frac{\partial V(z, t)}{\partial z} \tag{10a}
\]

\[
R(\bar{z}) = R(z)f'(\bar{z}), \quad L(\bar{z}) = L(z)f'(\bar{z}) \tag{10b}
\]

A. The Step Summing Coordinate Transformation

For convenience, let us work with the inverse relation \( \bar{z} = f^{-1}(z) \) instead of \( z = f(\bar{z}) \). Since a one-to-one mapping is required, this does not cause problems. Consider the coordinate transformation

\[
\bar{z} = f^{-1}(z) = z + \varphi(z), \quad \varphi(z) = \sum_{m=1}^{M} a_m d_m \varphi_m(z) \tag{11a}
\]

where

\[
a_m = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} - 1 \geq 0, \quad d_m = z_m - z_{m-1} + 20\Delta \bar{z}; \tag{11b}
\]

\[
\varphi_m(z) = \left[ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{\pi}{d_m}(z - (z_m + 10\Delta \bar{z})) \right) \right], \quad z_m - 10\Delta \bar{z} \leq z \leq z_m + 10\Delta \bar{z} \tag{11c}
\]

Parameters \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) were previously defined in (7) and (8). Also, \( \Delta \bar{z} \) represents the cell size of the uniform grid in \((\bar{z}, t)\)-space. This parameter must be chosen so that

\[
\lambda_{\text{max}}/20 \leq \Delta \bar{z} \leq \lambda_{\text{max}}/10. \tag{12}
\]

Once \( \Delta \bar{z} \) has been chosen, the total number of cells \( N_{\bar{z}} \)—which must be the same in both spatial domains—can be determined from

\[
N_{\bar{z}} = \text{round} \left( \frac{f^{-1}(z_M)}{\Delta \bar{z}} \right). \tag{13}
\]
Moreover, since renormalizing the per-unit-length parameters does achieve the desired mapping, observe that (11) yields \( \bar{z} = z \) in the main grid region (i.e. where \( a_m = 0 \)). This implies that the main grid will have identical cell sizes in both spatial domains. On the other hand, in the subgridding regions (i.e. where \( a_m > 0 \), the coordinate transformation adds a step function \( \varphi_m(z) \) that effectively contracts distances (by a factor \( 1/(a_m+1) \)) as we go from the \( \bar{z} \)-domain to the \( z \)-domain. This spatial contraction is what allocates a higher density of grid points in the subgridding regions even though a uniform grid is used in the \((\bar{z}, t)\)-space.

When solving a NTL problem using (11), three important practical issues must be kept in mind. First, in order to apply the transformation, we must know \( z = f(\bar{z}) \) at a uniformly spaced set of points \( \bar{z}_i = i \Delta \bar{z} \) for all \( i = 1, 2, \ldots, N_{\bar{z}} \). While it is possible to analytically invert (11) to obtain \( z_i = f(\bar{z}_i) \), the easiest way to obtain the \( z_i \)'s is through an interpolation routine that computes them from the data of a plot of \( f^{-1}(z) \) vs. \( z \). Second, in order to renormalize the per-unit-length parameters, it is necessary to obtain the renormalization factor \( f' (\bar{z}_i) \). Once the \( z_i \)'s have been obtained by interpolation, it is possible to obtain \( f' (\bar{z}_i) \) from

\[
\frac{1}{f'(z_i)} = \left( \frac{d}{d\bar{z}} f^{-1}(\bar{z}) \right)_{\bar{z}=z_i}.
\]

A closed form expression for the derivative of \( f^{-1}(z) \) can be easily obtained from (11).

**B. Stability Criterion**

The third and most important issue is the stability of the resulting FDTD scheme in the \((\bar{z}, t)\)-domain. As the per-unit-length parameters are renormalized, the stability criterion will be different from that shown in [3] for the conventional FDTD scheme. It can be shown that the proposed time stepping procedure is stable as long as

\[
\frac{\lambda_{max} f_{max} \Delta t}{\Delta \bar{z}} \leq S \leq 1, \quad S = \frac{z_0 + L_z}{z_0} \left\{ \frac{1}{f'(z)} \right\}.
\]

Moreover, since renormalizing the per-unit-length parameters does not affect the dual structure of the NTL equations, the proposed time stepping procedure does not suffer from late time instability problems [8].

**IV. SUBGRIDDING EXAMPLE: A BRAGG REFLECTOR**

To demonstrate how the proposed subgridding scheme works, consider the NTL structure shown in Fig. 1. The NTL is lossless with piecewise constant per-unit-length parameters given in Table I. This structure is the transmission-line equivalent of a Bragg reflector with two dielectric slabs. The goal in this example is to obtain \( f(\bar{z}) \) for this structure, and to compute the reflection coefficient at \( z = 8.622 \) cm assuming that absorbing boundaries are present at \( z = z_0 \) and \( z = z_2 \). This problem was solved in the \((\bar{z}, t)\)-domain by using a uniform discretization grid with the following parameters:

\[
\Delta \bar{z} = 1.268 \text{ mm} = \lambda_{max}/118.2, \quad \lambda_{max} = c_0/f_{max}
\]

\[
\Delta t = 0.09520 \Delta \bar{z}/c_0 = 0.4026 \text{ ps}, \quad f_{max} = 20 \text{ GHz}
\]

\[
N_{\bar{z}} = 424, \quad N_T = 21000.
\]

Both \( \Delta \bar{z} \) and \( N_{\bar{z}} \) were picked according to (12) and (13). Observe that if we were to solve the problem using a uniform discretization grid in the \((z, t)\)-domain with \( \Delta z = \lambda_{max}/118.2 \), the condition in (5) would be violated (since \( \lambda_{max} = 5 \lambda_{min} \)). To excite the NTL, a Gaussian pulse of the form

\[
f_S(t) = e^{-\left(\frac{t-t_0}{T_S}\right)^2}, \quad t_0 = 3T_S, \quad T_S = 1/(2f_{max})
\]

was added at every time step to the voltage node located at \( z = 6.340 \) cm. The voltage waveform was sampled at the node located at \( z = 8.622 \) cm in order to compute the reflection coefficients from a discrete Fourier transform of the incident and reflected pulses. The amplitude of the computed reflection coefficient is shown in Fig. 2 together with the corresponding analytic solution [9].

**A. The Coordinate System Transformation at Work**

Now that the reflection coefficient results have been presented, let us discuss how the coordinate transformation was applied to this problem. A plot of the generated coordinate transformation \( f(\bar{z}) \) is shown in Fig. 5. There are two features of this plot that are worth highlighting. First, in the regions where the slope of \( f(\bar{z}) \) is equal to one, the size of the grid cells in the \((z, t)\)-domain and the \((\bar{z}, t)\)-domain is identical. However, in regions where the slope of \( f(\bar{z}) \) is less than one, the coordinate transformation contracts distances as we go from the \( \bar{z} \)-domain to \( z \)-domain. As stated earlier, this contraction is what increases the density of grid points in the \( z \)-domain even though a uniform cell spacing is used in the \( \bar{z} \)-domain. Second, there are two regions where the slope of \( f(\bar{z}) \) is less than one. These two regions correspond to sections 2 and 4 of the NTL (see Fig. 4), which require subgrids with a density of grid points that is higher than in the rest of the structure. The subgrid that was introduced in the vicinity of section 2 is illustrated in Fig. 4. The drawn lines intersect at the voltage node locations in \((z, t)\)-space.
B. Performance of the Subgridding Scheme

The primary purpose of any subgridding scheme is to reduce the computational cost associated with solving the problem without compromising the accuracy of the solution. So, the next step in the discussion is to evaluate how the presented subgridding scheme performs compared to the conventional FDTD solution of the problem on a uniform grid. The same NTL problem was also solved using a uniform grid with the following parameters:

\[
\Delta z = 0.2536 \text{ mm} = \lambda_{min}/11.82, \quad \lambda_{max} = c_0/f_{max}
\]
\[
\Delta t = 0.4760 \Delta z/c_0 = 0.4026 \text{ ps}, \quad f_{max} = 20 \text{ GHz}
\]
\[
N_Z = 920, \quad N_T = 21000.
\]

Observe that, within sections 2 and 4 of the NTL, the uniform grid defined above has the same spatial resolution as the nonuniform grid previously discussed; however, the grid is unnecessarily fine within sections 1, 3 and 5 of the NTL.

The reflection coefficient results that were obtained using a uniform grid were found to be just as accurate as the reflection coefficient results previously presented in Fig. 2 for the nonuniform grid. A plot of the absolute value of the error in the reflection coefficient amplitude computation is shown in Fig. 5 for both the uniform and the nonuniform grids. In both cases, the error was defined as the absolute value of the difference between the computed reflection coefficient amplitude and the corresponding analytic solution. From the perspective of computational cost, introducing the subgridding scheme significantly reduced the computational effort required to solve the problem. The execution time for the implementation of the uniform grid solution was measured to be 5,580 sec whereas the execution time for the implementation of the nonuniform grid solution was measured to be 3,730 sec. This represents a 33\% reduction in the computational cost of the solution.

C. Spurious Reflections from the Subgrid Interfaces

Another issue that often arises when evaluating the performance of a subgridding scheme is the existence of spurious reflections at the interfaces between subgrids and the main grid. In the presented example, two subgrids were introduced in sections 2 and 4 of the NTL. In order to evaluate the spurious reflections that arise from the presence of these two subgrids, the reflection coefficient of the NTL was recomputed in the absence of the dielectric slabs (i.e. assuming \( L_m = \mu_0 \) and \( C_m = \varepsilon_0 \) for all \( m \)) keeping the nonuniform grid generated by the coordinate transformation shown in Fig. 2. The recomputed reflection coefficient is given in Fig. 6. Theoretically, no reflections should occur in the absence of the dielectric slabs; however, as shown in Fig. 6, the spurious reflections that do occur have a very small magnitude. As illustrated in Fig. 6, the proposed coordinate transformation introduces a smooth transition from the coarse main grid to the fine subgrids. This smooth transition ensures that the spurious reflections will have a very small amplitude.

V. CONCLUSION

In summary, a subgridding scheme for the FDTD solution of the NTL equations has been presented. A unique feature of this scheme is that it uses a coordinate transformation to introduce subgrids without having to give up the simplicity of the conventional FDTD time stepping algorithm. The presented coordinate transformation automatically selects an appropriate grid density for each region of the NTL based on the NTL’s material properties. In addition, the proposed subgridding scheme is stable and does not suffer from late time instability problems. Finally, computed results were presented to illustrate how distances can be manipulated to allocate local subgrids where they are needed.

REFERENCES