Dispersion Analysis and Comparative Study of Coifman scaling function based S-MRTD

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Abstract
A detailed analysis and comparative study of a finite difference numerical scheme (of the S-MRTD type) based on Coifman scaling functions (coiflets) of various degrees is presented in this paper. The smoothness properties of the Coifman basis functions provide for highly linear dispersion behavior of the resulting numerical scheme. In addition, the fact that coiflets are compactly supported adds to the numerical efficiency of the method. A comparison with Daubechies and Battle-Lemarie function based S-MRTD reveals a number of advantages in the use of coiflets as a basis for the formulation of S-MRTD schemes, while their overall performance is deemed similar to the former and better than the latter.

I. INTRODUCTION

The possibility to formulate time-domain numerical techniques with highly-linear dispersion properties, by employing smooth basis functions for the expansion of electromagnetic field components, was indicated in [1]. High dispersion linearity implies that a low discretization rate in space can still provide for a relatively small phase error in the numerical solution of a wave propagation problem. In addition, a dense grid (of the order of a tenth of a wavelength or less) corresponds to phase errors that are considerably smaller than those produced by the conventional Finite Difference Time Domain (FDTD) method. As a result, solving Maxwell’s equations in a relatively small domain can be accomplished with a coarse grid, while the accumulation of phase errors in multi-wavelength domains (for wireless communication or optical applications) can be limited without an excessive use of computational resources.

There have been several studies in the literature, regarding the dispersion properties of numerical schemes, developed along the lines of the formulation presented in [1], but employing different basis functions. Although a complete reference to the large volume of work that has been performed in this field would be impossible, it is worth mentioning papers on B-spline [2], Daubechies [3], [4] and biorthogonal [5] function based numerical schemes, as more relevant to this research. The motivation that drove the aforementioned authors to look for basis functions suitable for wave propagation problems, beyond the Battle-Lemarie cubic spline basis of [1] is clear: The “compact” support of the field expansion basis is always desirable, in order to limit the interaction of non-neighboring grid points and to facilitate the treatment of boundaries. It is worth noting, that the dominant term for such methods is the Multiresolution Time Domain (MRTD), although clearly no multiresolution property is carried by a single-level scheme. This terminology will be followed here, adopting the name of S-MRTD [1] for the proposed methods.

A compactly supported basis that has attracted a wide interest in the study of the frequency domain Method of Moments (MoM) and little attention for the construction of MRTD schemes, is the Coifman basis [6], [7], otherwise called coiflets. In particular, a coiflet-based MoM presented in [8], demonstrated the potential to achieve a significant MoM matrix sparsification. On the other hand, [9] presented a coiflet MRTD scheme applied to scattering problems, with quite promising results in terms of error and execution time. In this paper, a thorough study of coiflet-based S-MRTD is presented, along with a comparative study of the scheme, with respect to Daubechies and Battle-Lemarie S-MRTD. Several “degrees” of Coifman functions are considered (the term is explained later on) and dispersion curves are provided. Trade-offs between computational complexity, numerical error and grid size are investigated in detail, with the purpose of providing insights to the optimal choice of the parameters of the considered S-MRTD schemes. It is shown that coiflets can attain a similar numerical performance compared to Daubechies functions, while they present clear advantages over the Battle-Lemarie cubic-splines.

II. THE COIFMAN BASED S-MRTD SCHEME

Coifman systems are similar to Daubechies wavelets in the sense that they are derived by enforcing the condition that a number of their scaling function moments up to some degree are zero, but in

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addition to that, a number of their mother wavelet function $\psi$ moments up to some degree are zero as well. Explicitly, for a Coifman system of degree $N$, these conditions are [6], [7]:

$$\int_{-\infty}^{+\infty} x^k \phi(x) dx = \delta_k, \quad k = 0, 1, \ldots, N$$  \hspace{1cm} (1)

$$\int_{-\infty}^{+\infty} x^k \psi(x) dx = 0, \quad k = 0, 1, \ldots, N,$$  \hspace{1cm} (2)

with $\delta_k$ being Kronecker’s delta equal to 1, if $k = 0$, and zero otherwise. Representative plots of coi‡et scaling functions are provided in Fig. 1. It is noted that coi‡ets are compactly supported functions with excellent approximation properties, due to their smoothness. The corresponding S-MRTD scheme is readily derived via the well-known, Method of Moments based procedure that is outlined in [1]. Consider, for example, the $y$-component of Ampere’s law:

$$\varepsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$  \hspace{1cm} (3)

In the notation of [1], the S-MRTD finite difference form of (3) is:

$$n+1E_{y,i+\frac{1}{2},j} = nE_{y,i+\frac{1}{2},j} + \frac{\Delta t}{\varepsilon \Delta x} \sum_{p=-p_0}^{p_0-1} \alpha(p)n_{t+\frac{1}{2},i,j+p} \Delta H_x + \frac{\Delta t}{\varepsilon \Delta x} \sum_{p=-p_0}^{p_0-1} \alpha(p)n_{t+\frac{1}{2},i+p,j} \Delta H_y.$$  \hspace{1cm} (4)

The coefficients $\alpha(p)$ are the so-called stencil coefficients, given by the formula:

$$\alpha(p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lambda |\hat{\phi}(\lambda)|^2 \sin (\lambda(p+1/2))d\lambda,$$  \hspace{1cm} (5)

where $\hat{\phi}$ denotes a Fourier transform. The stencil $p_0$ is related to the effective support of the scaling function $\phi$ and despite the compact support of the coi‡ets can be considered as a parameter, in the sense of the Battle-Lemarie MRTD. That is, one does not have to include all the non-zero stencil coefficients in the update equations, but only those of them that are above a threshold value. However, it is worth noting that the decay of coi‡et stencil coefficients is much faster than the Battle-Lemarie coefficients. This fact is indicated by Fig. 2. As a result, the control of the numerical error stemming from the truncation of the finite difference operator in (4) is much easier than in Battle-Lemarie S-MRTD. It is also noted that the coefficients $\alpha(p)$ satisfy the symmetry condition $\alpha(-p) = -\alpha(p-1)$.

III. Dispersion Analysis and Comparative Study

The dispersion analysis of the Coifman based S-MRTD was carried out, via the standard Fourier dispersion analysis method. For a one-dimensional wave propagation within a uniform grid of cell size $\Delta$ and time step $\Delta t$, the dispersion equation assumes the form [10]:

$$\left(\frac{\Delta}{c\Delta t}\right)^2 \sin^2 \frac{\Omega}{2} = \left\{ \sum_{p=-p_0}^{p_0-1} \alpha(p) \sin(X(p+1/2)) \right\}^2,$$  \hspace{1cm} (6)
where \( X = k\Delta \) is the normalized wavenumber and \( \Omega = \omega \Delta t / \Delta \) the normalized frequency, with 
\( s = c \Delta t / \Delta \) being the CFL number. Fig. 3 depicts the dispersion curve for CFL number \( s = 0.065 \). 
Table I provides stability limits for several Coifman based schemes (denoted as \( C_n \), \( n \) being the degree of the Coifman basis). With the exception of \( C1 \)-MRTD, all the other Coifman schemes are characterized by a stability limit around 0.65. This also the case for Daubechies-based methods and the Battle-Lemarie S-MRTD. As a result, the curves of Fig. 3 correspond to approximately 0.1 of the stability limit for all Coifman based S-MRTD schemes. For comparison purposes, the theoretical and FDTD dispersion curves are appended. A very high linearity of the dispersion curve is observed, similar to the Battle-Lemarie S-MRTD of [1].

Next, the following study is carried out: The phase velocity error for Coifman based S-MRTD is parametrically studied and compared to Daubechies based S-MRTD and FDTD. The Daubechies S-MRTD dispersion was thoroughly investigated in [10]. Through this study, one can address the question what grid size each method requires in order to achieve a certain numerical error. Inspection of Fig. 4, indicates that Coifman based MRTD can achieve small phase errors at coarse grids. For example, \( C3 \)-S-MRTD allows for the choice of a nine times larger cell size than FDTD, an increase that fully compensates for the increase in operations per cell and the deterioration of the stability limit. Hence, Coifman based S-MRTD seems to be a good candidate for the accurate and efficient solution of large scale problems, involving multi-wavelength domains, addressing a well-known limitation of the conventional FDTD.

The error plot of Fig. 4 has been provided in a logarithmic scale, in order to emphasize that S-MRTD in general, is a second order scheme. This becomes clear by comparing the slopes of the error curves for S-MRTD and FDTD and noting that they are equal to 2. The reason for this effect is the use of a leap-frog time integration for the formulation of S-MRTD. It actually appears in all S-MRTD numerical schemes and is irrelevant to the particular choice of a Coifman basis. For that reason the dispersion performances of all Coifman based S-MRTD techniques seem to be almost identical, in agreement with the relevant studies on Daubechies schemes, included in [10]. On the contrary, the stencil sizes of these methods are largely different (see next section). Therefore, a low order Coifman based S-MRTD (for example \( C2, C4 \)) would be the method of choice, combining the favorable S-MRTD dispersion properties with a limited increase in the number of operations per cell (compared to FDTD).

IV. STENCIL-RELATED DISPERSION EFFECTS

The dependence of the phase velocity error on the number of stencil coefficients kept in the finite difference operator of Coifman-based S-MRTD is investigated. Thus, optimal stencil sizes for all schemes under study are computed. All error curves were derived for \( s = 0.065 \), in consistence with the ones that were presented earlier. It is noted that coefficients are compactly supported and as a result, their stencil is finite. Yet, this investigation shows that it is possible to reduce this finite stencil further (saving numerical operations), with an insignificant effect on numerical error. Figs. 5, 6 reveal the range of wavenumbers where each degree of Coifman scaling functions would produce a low-dispersion-error S-MRTD scheme, under a given choice of stencil size. It is shown that even a \( C2 \)-MRTD with stencil \( p_0 = 6 \) would be a suitable choice, with an error performance similar to a \( C7 \)-MRTD with stencil \( p_0 = 11 \). It is also observed that the numerical performance of the \( C9 \) scheme can be dramatically improved, by truncating its stencil size from 19 to 10, without any dramatic change in terms of error.

V. CONCLUSION

A thorough investigation of Coifman scaling function (of several degrees) based S-MRTD has been performed in this paper. Dispersion and stability analysis of this class of numerical schemes has demonstrated several advantages that they present, when applied to wave propagation problems. Comparison to the similar family of Daubechies compactly supported scaling function S-MRTD has led to the conclusion that coiflets can lead to similarly efficient numerical techniques. Despite the relatively long support of high degree coiflets, their stencil coefficients decay quite fast (much faster than the Battle-Lemarie stencil coefficients). As a result, truncation of the Coifman finite difference operator in space is possible with minimal effect on numerical errors.
Fig. 3. Dispersion curve for Coifman (of degrees 1-9) based S-MRTD, vs. theoretical and FDTD.

Fig. 4. Phase velocity error for Coifman (of degrees 1-9) based S-MRTD.

Fig. 5. Stencil effect on Coifman based S-MRTD of degrees 1-9 (C_n − m means that a degree n Coifman S-MRTD with stencil size m is used).

Fig. 6. Stencil effect on Coifman based S-MRTD of degrees 6, 9 (C_n − m means that a degree n Coifman S-MRTD with stencil size m is used).

REFERENCES