A Spatial Filter-Enabled High-Resolution Subgridding Scheme for Stable FDTD Modeling of Multiscale Geometries

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Abstract—As many real-life microwave structures include multiple space and time scales, research on stable subgridding schemes is as timely as ever. This paper presents a new approach to this old problem, within the context of the FDTD technique, based on a spatial filtering method that enables dense and coarse grids to be run at the time-step of the coarse grid. A spatial filter, coupled to the FDTD update equations, stabilizes those subgrids that are simulated at time steps above their stability limit. The proposed method is used to implement very high aspect ratio subgrids, maintaining late-time stability over millions of time steps.

Index Terms—FDTD methods, numerical stability, digital filters.

I. INTRODUCTION

As the complexity of microwave circuits and systems grows, electromagnetic simulators are required to efficiently handle multiscale geometries. For time-domain methods, this translates to the dual challenge of space and time mesh refinement, typically hampered by spurious reflections at the dense/coarse grid interfaces and the problem of late-time instability. In the context of the Finite-Difference Time-Domain (FDTD) method several approaches have been presented [1]-[5]. When mesh refinement in space is performed, the Courant stability criterion limits the time step throughout the mesh as:

$$\Delta t \leq \sqrt{\frac{1}{\Delta x_{min}^2} + \frac{1}{\Delta y_{min}^2} + \frac{1}{\Delta z_{min}^2}}$$

(1)

where $$\Delta x_{min}$$ are the dimensions of the smallest cell in the $$\xi$$-direction, $$\xi = x, y, z$$. This motivates a parallel effort to apply different time steps within each subgrid, in order to surpass the limitation imposed by (1). However, early or late-time instability typically occurs as a result [3], [5].

This paper explores an alternative way to alleviate the computational burden stemming from (1) in multiscale simulation scenarios, without incurring late-time instability. The key idea, recently explored in [6], is to use a spatial filter to enable all subgrids to use the same time step, that time step not being limited by (1). The role of the filter is to eliminate unstable spatial harmonics that are excited when (1) is violated, hence reinforcing stability. Notably, digital signal processing techniques have been previously employed for various purposes in FDTD. For example, [5] used a single-pole filter to delay the occurrence of late-time instability. Moreover, [7] described a spatial filtering approach to minimize dispersion errors. Similarly, spatial filters were utilized in [8] to suppress spurious reflections at subgrid interfaces. However, in all these schemes, FDTD was operated at a Courant-limit compatible time step. On the other hand, the present work employs a spatial filter to overcome the limitation of (1) for FDTD, similar to earlier work in the area of computational fluid dynamics pertinent to explicit schemes for the Navier-Stokes equations [9].

The rest of the paper briefly reviews the spatial filtering algorithm of [6] and proceeds with its application to subgridding. A benchmark test case of a subgrid embedded in a cavity is used to characterize the accuracy and performance of the algorithm, with emphasis on the demonstration of its late-time stability over two millions of time-steps. Finally, the proposed methodology is applied to a parallel plate waveguide loaded with two diaphragms, where subgrids are placed around the slits of the diaphragms.

II. OVERCOMING THE FDTD STABILITY LIMIT WITH SPATIAL FILTERING

The standard procedure for proving (1) consists of the following steps [10]. First, the dispersion analysis of a three-dimensional system of FDTD equations results in the condition:

$$\sin \frac{\omega \Delta t}{2} = \frac{\Delta t}{\sqrt{\epsilon \mu}} \sqrt{\frac{\sin^2 \tilde{k}_x \Delta x}{\Delta x^2} + \frac{\sin^2 \tilde{k}_y \Delta y}{\Delta y^2} + \frac{\sin^2 \tilde{k}_z \Delta z}{\Delta z^2}}$$

(2)

with $$\tilde{k} = \tilde{x} \tilde{k}_x + \tilde{y} \tilde{k}_y + \tilde{z} \tilde{k}_z$$ being the numerical wavevector. For (2) to yield real frequencies $$\omega$$, the right-hand-side of the equation has to be less than one. As the sinusoidal terms can be at most one, (1) guarantees that indeed $$\sin(\omega \Delta t/2) \leq 1$$ for any wavevector $$\tilde{k}$$. The sinusoidal terms in (2) can indeed take any value in $$[-1, 1]$$, necessitating the enforcement of stability over their whole range. Nevertheless, numerical dispersion errors become prohibitively large if less than ten grid points per simulated wavelength are used. Therefore, both the cell size and the source excitation are chosen in consistency with this condition, essentially suppressing the useful part of the spatial frequency spectrum to wavevectors with $$|\tilde{k}| \Delta < 0.2\sqrt{3}\pi$$, where $$\Delta x = \Delta y = \Delta z = \Delta$$ to simplify the formulation. The rest of the spectrum is weakly excited and contributes to the error of the solution, as long as it remains stable, which is why (1) is needed. Depending on the problem at hand and its required mesh density, the range
of “useful” wavevectors may be even further reduced. Let $X_{max} = \| \tilde{\kappa} \|_{max} \Delta$ be the maximum normalized wavenumber within this range. Then, for $\xi = x, y, z$:

$$\tilde{k}_\xi \leq \| \tilde{\kappa} \|_{max}$$

(3)

and:

$$\sqrt{\sin^2 \frac{k_x \Delta}{2} + \sin^2 \frac{k_y \Delta}{2} + \sin^2 \frac{k_z \Delta}{2}} < \sqrt{3} \sin \frac{X_{max}}{2}$$

(4)

The latter implies that (2) admits real solutions with respect to $\omega$ if:

$$\Delta t \leq \frac{\Delta}{\sqrt{3} \sin \frac{X_{max}}{2}}$$

(5)

or, namely, the extension of the conventional Courant limit by a “Courant limit enhancement” factor $CE$:

$$CE = 1 / \sin \frac{X_{max}}{2}$$

(6)

The prerequisite to realize this enhancement of the Courant limit is to render the FDTD waveforms spectrally bounded. As the FDTD domain is spatially limited, this can only be achieved by the iterative application of a spatial filter that would cut-off spatial frequencies beyond $\| \tilde{\kappa} \|_{max}$:

$$\overline{F}(\tilde{\kappa}) = \begin{cases} 1, & \text{for } \sqrt{k_x^2 + k_y^2 + k_z^2} \leq \| \tilde{\kappa} \|_{max} \\ 0, & \text{otherwise} \end{cases}$$

(7)

The application of this filter should have a controllable impact on the accuracy of FDTD, since the rejected spatial frequencies are anyway erroneous, unwanted and close to the error floor of the solution. Note that any smooth filter that would attenuate instead of removing unstable spatial harmonics is bound to result in early or late-time instability. To our experience and knowledge, (7) is the only viable filter.

The spatial filtering process is carried out in the spectral domain. In particular, once the electric fields have been updated, they are transformed into the $\tilde{\kappa}$ domain via a Fast Fourier transform (FFT). The $\tilde{\kappa}$-components outside the pass-band are set to zero. Subsequently, an inverse Fourier transform in applied to retrieve the electric field in the space domain. A similar process is applied to the magnetic field. These steps represent the overhead needed in order to extend the FDTD stability limit. The cost for implementing these steps can be minimized by efficient implementation of the FFT, in a similar fashion to the PSTD technique [11], and parallelization either using multiple processors or Graphics Processor Units. In many cases, it is simply offset by the increase in the time step achieved, as shown in [6]. Moreover, the additional FFT operations represent small modifications of the standard FDTD algorithm. This feature of the spatially filtered FDTD is in stark contrast with alternative methods to overcome (1), such as the Alternating Direction Implicit (ADI)-FDTD method which is based on a totally different time integration scheme. Finally, if applied locally as in a subgrid, the spatially filtered FDTD can enable stable mesh refinement, resulting in substantial computational gains in multiscale problems. This point is further explored in the next sections.

III. SPATIALLY FILTERED FDTD BASED SUBGRIDDING: ALGORITHM AND VALIDATION

A. Algorithm

The proposed subgridding technique couples the spatially filtered FDTD with the late-time stable algorithm of [4]. While the subgrid of [4] needs to be run at a time step bounded by the Courant limit of the densest grid, spatial filtering removes this limitation. To this end, a single time step of the algorithm consists of the next sub-steps, for an odd mesh refinement ratio $N_R$:

1) Coarse grid electric fields within the coarse grid and on the boundary of coarse/dense grid are updated regularly.
2) Dense grid electric fields in the outermost $(N_R+1)/2$ dense cells of the subgrid (into the coarse grid) are assigned the value of the corresponding and overlapping coarse electric fields (right on the boundary of coarse/dense grids). The remaining interior electric field components in the dense grid are updated and spatially filtered.
3) Dense grid magnetic field components are updated and spatially filtered throughout the dense grid.
4) The coarse grid magnetic field components within the coarse grid are updated regularly, whereas the coarse grid magnetic field components one coarse cell into the subgrid are assigned the average value of the corresponding $N_R$ dense grid magnetic fields that are defined in that cell.

Note that for the main grid and the subgrid to be connected through boundary conditions, the extension of the one grid into the other by about one coarse grid cell is necessary.

B. Validation and Benchmarking

![Fig. 1. A subgrid embedded in a square metallic cavity.](image)

The spatial filter-enabled FDTD subgridding method is tested with the computation of the TM resonant modes of a square cavity on the $x-y$ plane, where a subgrid is embedded as shown in Fig. 1. In the main (coarse) grid, the Yee cell dimensions are $\Delta x_c = \Delta y_c = \Delta z = 5$ cm. The cavity is $1.4m \times 1.4m$, hence including $28 \times 28$ coarse grid cells. A
0.5m×0.5m area in the middle of the cavity is covered by the subgrid. Within the subgrid, the mesh is refined by an odd factor $N_R$. In the following, results for mesh refinement factors $N_R=3, 11, 29$ are shown. Regardless of the mesh refinement ratio, the same time step is applied in both meshes:

$$\Delta t = 0.95\sqrt{\epsilon_0\mu_0}\frac{\Delta c}{\sqrt{2}}$$  \hspace{1cm} (8)

Hence, the dense grid is run at a factor

$$s_D = \frac{\Delta t}{\sqrt{\epsilon_0\mu_0}\frac{\Delta c}{\sqrt{2}N_R}} \approx 0.95N_R$$  \hspace{1cm} (9)

times its own Courant stability limit, with $s_D=2.85$, 10.45 and 27.55, respectively, stabilized by the spatial filter that is parameterized by $\chi = |\mathcal{F}|_{max}\Delta c/N_R$. For the three mesh refinement cases under study, $\chi$ is set to 0.9353, 0.2551 and 0.0968, respectively.

In all simulations, 40,000 coarse grid time steps are used. Table I depicts the simulated resonant frequencies for four TM modes, along with their relative errors with respect to their analytically calculated values and the associated execution time. Moreover, the execution time of an FDTD scheme, with the corresponding dense grid extended throughout the cavity, is appended. In all cases, very small errors have been found, even though the mesh refinement ratios reported are very large compared to those typically negotiated in the subgridding literature. The Courant limit of the subgrid is also significantly exceeded within the subgrid (up to 27.55 times).

| Table I |

| Resonant Frequencies [in GHz], Relative Errors (R.E.) [in %] and Execution Time [in sec] for the Spatial Filter-Enabled Subgridded FDTD |
|---|---|---|---|---|
| Mode | $N_R$ | $s_D$ | $\chi$ (rad) | Time (sec) |
| TM1.1 | 3 | 2.85 | 0.9353 | 347 |
| | 11 | 10.45 | 0.2551 | 2672 |
| | 29 | 27.55 | 0.0968 | 48960 |
| TM2.2 | 3 | 0.302916 | +0.026 | 527 |
| | 11 | 0.302916 | +0.026 | 2672 |
| | 29 | 0.302916 | +0.026 | 48960 |
| TM3.2 | 3 | 0.38925 | -0.030 | 527 |
| | 11 | 0.38925 | -0.030 | 2672 |
| | 29 | 0.38925 | -0.030 | 48960 |
| TM3.4 | 3 | 0.535318 | -0.005 | 527 |
| | 11 | 0.535318 | -0.005 | 2672 |
| | 29 | 0.535318 | -0.005 | 48960 |

Fig. 2. Demonstration of the stable late-time behavior of the fields in the cavity of Fig. 1, determined by the spatial filter-enabled subgridded FDTD, over two million time steps. The magnetic field component of the TM$_{1,1}$ mode is plotted for $N_R = 3$.

Finally, the magnetic field component $H_z$ is plotted for the TM$_{1,1}$ and TM$_{3,4}$ modes for $N_R = 11, 29$ in Fig. 3. Relatively smooth mode profiles are shown in all cases, again unaffected by the extreme refinement ratios within the subgrid and the fact that the subgrid is operated well beyond its Courant limit.

The late-time stability of the code has been carefully studied. Over millions of time-steps, no late-time instability has been observed. As an example, the temporal waveform of the magnetic field $H_z$ is plotted in Fig. 2 for the last 20,000 steps until the completion of two million time steps. No growth or dissipation (which would have been more possible, given that the filter iteratively removes spatial frequencies from the solution) can be observed.

Fig. 3. TM$_{1,1}$ and TM$_{3,4}$ mode profiles ($H_z$) extracted with the spatial filter-enabled subgridded FDTD for $N_R=11, 29$.

IV. SPATIALLY FILTERED FDTD BASED SUBGRIDDING: APPLICATION

The performance and versatility of the proposed method is demonstrated with an application requiring two subgrids embedded in a coarse mesh. The structure under study is a parallel-plate waveguide with two diaphragms, as shown in Fig. 4. The width of the slits in Fig. 4 is 1.667 cm, while the coarse grid consists of Yee cells with $\Delta x_c = \Delta y_c = \Delta z_c = 5$ cm.

To enhance the capability of the spatially-filtered FDTD subgrid to account for the edge singularities introduced by the presence of the slits, both subgrids are divided in two sections, along their respective diaphragm. Spatial filtering is applied to...
each section, i.e. to four sections in total. Note that within each section, not only the $y-$ but also the $x-$ components of the electric field are continuous. This would not have been the case, had the subgrids not been divided.

The results shown in Fig. 5 are based on the proposed method with $N_R=3$, compared to a reference FDTD simulation where the subgrid cell size has been used throughout the computational domain. The time step used throughout the mesh for this paper’s method corresponds to $1.5$ times the Courant limit of the subgrid (stabilized by a filter with $\chi = 1.7031$ rad) and $0.5$ of the Courant limit of the coarse grid. Also, results derived using the subgridding algorithm of [4] are shown, under the same meshing conditions, but with the time step being $0.5$ of the Courant limit of the dense grid. A soft Gaussian source of the form $\exp(-t-t_0)^2/T_s^2$, with $T_s=1179$ ps and $t_0=4717$ ps, is used to excite the magnetic field $H_z$.

The time-domain waveform of $H_z$ is probed $47.5$ cm past the right slit, and the $S_{21}$ of the system of the two slits is calculated. Notably, this work leads to more accurate results, compared to [4], an indication that spatial filtering not only stabilized the subgrid, but also filtered out spurious reflections stemming from the dense to coarse grid interface. Being free from the Courant stability limit of the dense grid, the spatial filter-enabled subgrid took 1083 sec, as opposed to 2824 sec for the subgridding technique of [4], over the time span of 15,000 time steps of the coarse grid.

V. CONCLUSIONS

Based on a spatially-filtered FDTD method, a new subgridding approach has been presented. With a classical benchmarking study of a subgrid embedded in a metallic cavity, it has been shown that this approach can lead to stable, high-resolution mesh refinement. Moreover, the restriction on the time step that is imposed by the Courant stability limit of the dense grid has been waived via spatial filtering, without introducing late-time instability. Hence, the proposed technique is significantly faster than alternative late-time subgridding methods, which are invariably subject to the aforementioned limitation.

REFERENCES